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## SPECTRAL RADII OF GENERALIZED INVERSES OF SIMPLY POLAR MATRICES

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**ABSTRACT.** In this note we study the spectral radii of generalized inverses of square matrices  $A$  such that  $\text{rank}(A) = \text{rank}(A^2)$ .

### 1. General and introductory material

For positive integers  $n$  and  $m$ ,  $\mathbb{C}^{n \times m}$  denotes the vector space of all complex  $n \times m$  matrices.

Let  $A \in \mathbb{C}^{n \times n}$  be a square matrix. A matrix  $C \in \mathbb{C}^{n \times n}$  is called a  $g_1$ -inverse of  $A$  if

$$ACA = A.$$

If  $B \in \mathbb{C}^{n \times n}$  and

$$ABA = A \quad \text{and} \quad BAB = B,$$

then  $B$  is called a  $g_2$ -inverse of  $A$ . By  $\mathcal{G}_1(A)$  we denote the set of all  $g_1$ -inverses of  $A$ .  $\mathcal{G}_2(A)$  is the set of all  $g_2$ -inverses of  $A$ . It is well-known that  $\mathcal{G}_1(A) \neq \emptyset$  (see [1]). Furthermore it is easy to see that if  $C \in \mathcal{G}_1(A)$ , then  $B = CAC \in \mathcal{G}_2(A)$ , hence

$$\emptyset \neq \mathcal{G}_2(A) \subseteq \mathcal{G}_1(A).$$

If  $A$  is non-singular, then  $\mathcal{G}_2(A) = \mathcal{G}_1(A) = \{A^{-1}\}$ .

For  $A \in \mathbb{C}^{n \times n}$  we denote the set of eigenvalues of  $A$  by  $\sigma(A)$  and the *spectral radius*  $r(A)$  of  $A$  is defined by

$$r(A) = \max_{\lambda \in \sigma(A)} |\lambda|.$$

Let  $A \in \mathbb{C}^{n \times m}$ .  $A^T$  denotes the transpose of  $A$  and  $A^*$  denotes the conjugate transpose of  $A$ . The *range* of  $A$  is given by

$$\mathcal{R}(A) = \{Ax : x \in \mathbb{C}^n\}$$

and the *kernel* of  $A$  is the set

$$\mathcal{N}(A) = \{x \in \mathbb{C}^n : Ax = 0\}$$

(we follow the convention  $\mathbb{C}^n = \mathbb{C}^{n \times 1}$ ).

In this note we study the set

$$R_A = \{r(C) : C \in \mathcal{G}_1(A)\}$$

for  $A \in \mathbb{C}^{n \times n}$  such that  $\text{rank}(A) = \text{rank}(A^2)$ , where  $\text{rank}(A) = \dim \mathcal{R}(A)$ . Such matrices are called *simply polar*.

**Examples.** If  $A$  is non-singular, then  $R_A = \{r(A)^{-1}\}$ . If  $A = 0$ , then  $ACA = A$  for each  $C \in \mathbb{C}^{n \times n}$ , hence  $R_A = [0, \infty)$ .

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Throughout this paper we will assume that  $n \geq 2$ . The identity on  $\mathbb{C}^n$  is denoted by  $I_n$ .

**1.1. Proposition.** If  $A \in \mathbb{C}^{n \times n}$  and  $B \in \mathcal{G}_2(A)$ , then

$$\mathcal{G}_1(A) = \{B + T - BATAB : T \in \mathbb{C}^{n \times n}\}$$

*Proof.* [1, Theorem 2 in Chapter 2.3]. □

It follows from Proposition 1.1, that if  $A$  is singular, then  $\mathcal{G}_1(A)$  is an infinite set. In [6], the following result is shown:

**1.2. Proposition.** Suppose that  $A \in \mathbb{C}^{n \times n}$  is singular. We have:

(1) for each  $z \in \mathbb{C}$ , there is  $B \in \mathcal{G}_1(A)$  with  $z \in \sigma(B)$ ;

(2) if  $B \in \mathcal{G}_2(A)$ , then

$$B + z(I_n - BA), B + z(I_n - AB) \in \mathcal{G}_1(A)$$

for all  $z \in \mathbb{C}$  and

$$r(B + z(I_n - BA)) = r(B + z(I_n - AB)) = \begin{cases} r(B), & \text{if } |z| \leq r(B) \\ |z|, & \text{if } |z| > r(B); \end{cases}$$

(3)  $[r(B), \infty) \subseteq R_A$  for each  $B \in \mathcal{G}_2(A)$ .

**1.3. Proposition.** Suppose that  $A \in \mathbb{C}^{n \times n}$ ,  $r = \text{rank}(A) > 0$  and that  $A$  has a decomposition

$$A = U \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} V^{-1}$$

with  $U, V \in \mathbb{C}^{n \times n}$  non-singular and  $D \in \mathbb{C}^{r \times r}$  non-singular. Then

$$B = V \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^{-1} \in \mathcal{G}_2(A)$$

and

$$\mathcal{G}_1(A) = \left\{ V \begin{bmatrix} D^{-1} & A_1 \\ A_2 & A_3 \end{bmatrix} U^{-1} : A_1 \in \mathbb{C}^{r \times (n-r)}, A_2 \in \mathbb{C}^{(n-r) \times r}, A_3 \in \mathbb{C}^{(n-r) \times (n-r)} \right\}.$$

*Proof.* It is easy to verify that  $B \in \mathcal{G}_2(A)$ . Let  $T \in \mathbb{C}^{n \times n}$ , let  $\varphi(T) = V^{-1}TU$  and set  $B_0 := B + T - BATAB$ . Then

$$\begin{aligned} B_0 &= V \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^{-1} + T - V^{-1} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \varphi(T) \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} U^{-1} \\ &= V \left( \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \varphi(T) - \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \varphi(T) \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \right) U^{-1} \\ &= V \begin{bmatrix} D^{-1} & A_1 \\ A_2 & A_3 \end{bmatrix} U^{-1}. \end{aligned}$$

Since the mapping  $\varphi : \mathbb{C}^{n \times n} \rightarrow \mathbb{C}^{n \times n}$  is bijective, the result follow from Proposition 1.1. □

Recall that a matrix  $A \in \mathbb{C}^{n \times n}$  is called *simply polar* if  $\text{rank}(A) = \text{rank}(A^2)$ .

**1.4. Proposition.** *Let  $A \in \mathbb{C}^{n \times n}$  be singular. The following assertions are equivalent:*

- (1)  $A$  is simply polar;
- (2)  $0$  is a simple pole of the resolvent  $(\lambda I_n - A)^{-1}$ ;
- (3)  $\mathbb{C}^n = \mathcal{R}(A) \oplus \mathcal{N}(A)$ ;
- (4) there is  $B \in \mathcal{G}_2(A)$  such that  $AB = BA$ .

*Proof.* [3, Satz 72.4], [3, Satz 101.2] and [1, Theorem 5.2]. □

If  $A \in \mathbb{C}^{n \times n}$  is simply polar, then, by Proposition 1.4, there is  $B \in \mathbb{C}^{n \times n}$  such that  $ABA = A$ ,  $BAB = B$  and  $AB = BA$ . It is shown in [1, Theorem 5.1], that there is no other  $g_2$ -inverse of  $A$  which commutes with  $A$ .  $B$  is called the *Drazin-inverse* of  $A$ . The following result is shown in [1, p. 53].

**1.5. Proposition.** *If  $A \in \mathbb{C}^{n \times n}$ ,  $A \neq 0$  and if  $A$  is simply polar, then the Drazin-inverse  $B$  of  $A$  satisfies*

$$\sigma(B) \setminus \{0\} = \left\{ \frac{1}{\lambda} : \lambda \in \sigma(A) \setminus \{0\} \right\}$$

*and hence  $r(B) = r(A)^{-1}$ .*

## 2. Generalized inverses of simply polar matrices

*Throughout this section we assume that  $A \in \mathbb{C}^{n \times n}$  is simply polar and that  $\text{rank}(A) > 0$ .*

By [5, 4.3.2 (4)] (see also [4]),  $A$  has a decomposition

$$(2.1) \quad A = U \begin{bmatrix} D & \vdots & 0 \\ \cdots & \vdots & \cdots \\ 0 & \vdots & 0 \end{bmatrix} U^{-1},$$

where  $U \in \mathbb{C}^{n \times n}$  and  $D \in \mathbb{C}^{r \times r}$  are non singular. From Proposition 1.3 we know that

$$(2.2) \quad B = U \begin{bmatrix} D^{-1} & \vdots & 0 \\ \cdots & \vdots & \cdots \\ 0 & \vdots & 0 \end{bmatrix} U^{-1} \in \mathcal{G}_2(A).$$

It is easy to see that the matrix  $B$  in (2.2) is the Drazin-invers of  $A$ .

**2.1. Theorem.** *The following assertions are equivalent:*

- (1)  $\dim \mathcal{N}(A) \geq \text{rank}(A)$ .
- (2) there is  $B \in \mathcal{G}_2(A)$  with  $B^2 = 0$ .

A consequence of Theorem 2.1 is:

**2.2. Corollary.** *If  $\dim \mathcal{N}(A) \geq \text{rank}(A)$ , then there is an entire function  $F : \mathbb{C} \rightarrow \mathbb{C}^{n \times n}$  such that*

$$F(z) \in \mathcal{G}_1(A), \sigma(F(z)) = \{z, 0\} \text{ and } r(F(z)) = |z| \text{ for all } z \in \mathbb{C}.$$

*Furthermore we have  $R_A = [0, \infty)$ .*

*Proof.* By Theorem 2.1, there is  $B \in \mathcal{G}_2(A)$  with  $B^2 = 0$ . Define  $F$  by  $F(z) = B + z(I_n - AB)$ . Then  $F(z) \in \mathcal{G}_1(A)$  for each  $z \in \mathbb{C}$  (Proposition 1.2). [6, Theorem 3] gives

$$\{z\} \subseteq \sigma(F(z)) \subseteq \{z, 0\} \quad (z \in \mathbb{C}).$$

Assume that  $F(z)$  is non-singular for some  $z \in \mathbb{C}$ . Thus there is  $C \in \mathbb{C}^{n \times n}$  with  $F(z)C = I_n$ . Since  $BF(z) = 0$ , we get  $0 = BF(z)C = B$ , thus  $A = ABA = 0$ , a contradiction. □

*Proof of Theorem 2.1.* Let  $r = \text{rank}(A)$ .

(1)  $\Rightarrow$  (2): Proposition 1.4 (3) shows that  $n - r = \dim \mathcal{N}(A) \geq r$ .

*Case 1:*  $n - r = r$ . Let  $D$  be as in (2.1) and let

$$S = \begin{bmatrix} D^{-1} & \vdots & D^{-1} \\ \cdots & \cdots & \cdots \\ -D^{-1} & \vdots & -D^{-1} \end{bmatrix} \text{ and } B = USU^{-1}.$$

Then it is easy to see  $B \in \mathcal{G}_2(A)$  and  $B^2 = 0$ .

*Case 2:*  $n - r > r$ . Then  $r < n/2$ .

*Case 2.1:*  $n = 2m$  for some  $m \in \mathbb{N}$ . Let

$$T = \begin{bmatrix} D^{-1} & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & 0 \end{bmatrix} \in \mathbb{C}^{m \times m}, S = \begin{bmatrix} T & \vdots & T \\ \cdots & \cdots & \cdots \\ -T & \vdots & -T \end{bmatrix} \in \mathbb{C}^{n \times n}$$

and  $B = USU^{-1}$ . Then  $B \in \mathcal{G}_2(A)$  and  $B^2 = 0$ .

*Case 2.2:*  $n = 2m + 1$  for some  $m \in \mathbb{N}$ . Then  $r < m$ . Set

$$T = \begin{bmatrix} D^{-1} & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & 0 \end{bmatrix} \in \mathbb{C}^{m \times m}, S = \begin{bmatrix} T & \vdots & T & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \vdots \\ -T & \vdots & -T & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \vdots \\ 0 & \cdots & 0 & \vdots & 0 \end{bmatrix} \in \mathbb{C}^{n \times n}$$

and  $B = USU^{-1}$ . As above,  $B \in \mathcal{G}_2(A)$  and  $B^2 = 0$ .

(2)  $\Rightarrow$  (1): Since  $B^2 = 0$ ,  $A$  is singular. We have  $(BA)^2 = BA$ ,  $\mathcal{R}(BA) = \mathcal{R}(B)$ ,  $\mathcal{N}(A) = \mathcal{R}(I - BA)$ ,  $\mathcal{R}(AB) = \mathcal{R}(A)$ ,  $(AB)^2 = AB$  and

$$\mathbb{C}^n = \mathcal{R}(B) \oplus \mathcal{N}(A),$$

thus, by Proposition 1.4 (3),  $\text{rank}(B) = r = \text{rank}(A)$ . Now let  $z \in \mathcal{R}(A) \cap \mathcal{R}(B)$ . Then  $z = ABz = BAz$ , therefore  $z = AB^2Az = 0$ . This gives  $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$ . Since

$$\mathcal{R}(A) \oplus \mathcal{R}(B) \subseteq \mathbb{C}^n,$$

we derive  $2r \leq n$ , hence  $\text{rank}(A) = r \leq n - r = \dim \mathcal{N}(A)$ .  $\square$

A square matrix  $D$  is said to be *non-derogatory* if its characteristic polynomial is also its minimal polynomial.

**2.3. Theorem.** Suppose that  $\text{rank}(A) = \text{rank}(A^2) = n - 1$ , let  $D$  be as in (2.1) and suppose that  $D$  is non-derogatory. Then  $A$  has a nilpotent  $g_1$ -inverse and hence  $\min R_A = 0$ .

*Proof.* Since  $D^{-1}$  is also non-derogatory, it follows from [2, Theorem 3.4] that there are  $a_1 \in \mathbb{C}^{n-1}$ ,  $a_2 \in \mathbb{C}^{n-1}$  and  $a_3 \in \mathbb{C}$  such that

$$S = \begin{bmatrix} D^{-1} & \vdots & a_1 \\ \cdots & \cdots & \cdots \\ a_2^T & \vdots & a_3 \end{bmatrix} \text{ is nilpotent,}$$

hence  $S^q = 0$  for some positive integer  $q$ . Let  $B = USU^{-1}$ . Then  $B^q = 0$ . By Proposition 1.3,  $B \in \mathcal{G}_1(A)$ .  $\square$

A matrix  $N \in \mathbb{C}^{n \times n}$  is called *normal* if  $NN^* = N^*N$ . The spectral theorem for normal matrices implies that

$$(2.3) \quad N = U \begin{bmatrix} D & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & 0 \end{bmatrix} U^*,$$

with  $U \in \mathbb{C}^{n \times n}$  unitary (that is  $UU^* = U^*U = I_n$ ) and  $D = \text{diag}(\lambda_1, \dots, \lambda_r)$ , where  $\lambda_1, \dots, \lambda_r$  are the non-zero eigenvalues of  $N$ . It follows (see [5, 4.3.2 (4)]) that  $N$  is simply polar.

Now suppose that  $\text{rank}(N) = n - 1$ . If  $\lambda_i \neq \lambda_j$  ( $i \neq j; i, j = 1, \dots, n - 1$ ) then the matrix  $D$  in (2.3) is

non-derogatory.

Thus we have proved:

**2.4. Corollary.** *If  $N \in \mathbb{C}^{n \times n}$  is normal,  $\text{rank}(N) = n - 1$  and if  $\lambda_i \neq \lambda_j$  ( $i \neq j; i, j = 1, \dots, n - 1$ ) for the non-zero eigenvalues of  $N$ , then there is a nilpotent  $g_1$ -inverse of  $A$ .*

### 3. The case $n = 2$

**3.1. Proposition.** *If  $A \in \mathbb{C}^{2 \times 2}$  and  $A^2 = 0$ , then there is  $B \in \mathcal{G}_2(A)$  such that  $B^2 = 0$ .*

*Proof.* The Schur decomposition of  $A$  is

$$A = U \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix} U^*,$$

where  $U \in \mathbb{C}^{2 \times 2}$  is unitary and  $\alpha \in \mathbb{C}$  (see [5, 5.2.3 (1)]). If  $\alpha = 0$ , we are done. So assume that  $\alpha \neq 0$ . Let

$$B = U \begin{bmatrix} 0 & 0 \\ \alpha^{-1} & 0 \end{bmatrix} U^*.$$

then it is easy to see that  $B \in \mathcal{G}_2(A)$  and  $B^2 = 0$ .  $\square$

**3.2. Theorem.** *Suppose that  $A \in \mathbb{C}^{2 \times 2}$  is singular. Then there is  $B \in \mathcal{G}_2(A)$  with  $B^2 = 0$  and hence  $R_A = [0, \infty)$ .*

*Proof.* Because of Proposition 3.1, we assume that  $A^2 \neq 0$ . Since  $A$  is singular, we have  $\text{rank}(A) = \text{rank}(A^2) = 1$ ,  $A$  is simply polar and  $\dim \mathcal{N}(A) = \text{rank}(A)$ . Theorem 2.1 gives the result.  $\square$

### 4. Generalized inverses of projections

In this section we assume that  $P \in \mathbb{C}^{n \times n}$ ,  $0 \neq P \neq I_n$  and  $P^2 = P$ . Hence  $P$  is simply polar.

Since  $\mathcal{R}(P) = \{x \in \mathbb{C}^n : Px = x\}$ , it follows that  $\sigma(P) = \{0, 1\}$  and that there is a non-singular  $U \in \mathbb{C}^{n \times n}$  such that

$$(4.1) \quad P = U \begin{bmatrix} I_r & \vdots & 0 \\ \cdots & \vdots & \cdots \\ 0 & \vdots & 0 \end{bmatrix} U^{-1}$$

([5, 9.8 (3)]), where  $r = \text{rank}(P)$ .

From Theorem 2.1 we know that

$$\dim \mathcal{N}(P) \geq \text{rank}(P) \Leftrightarrow \text{there is } B \in \mathcal{G}_2(P) \text{ such that } B^2 = 0.$$

So it remains to investigate the case where  $\dim \mathcal{N}(P) < \text{rank}(P)$ :

**4.1. Theorem.** *If  $\dim \mathcal{N}(P) < \text{rank}(P)$  and if  $B \in \mathcal{G}_1(P)$ , then  $1 \in \sigma(B)$  and hence  $r(B) \geq 1$ .*

*Proof.* Proposition 1.3 and (4.1) show that there are  $A_1 \in \mathbb{C}^{r \times (n-1)}$ ,  $A_2 \in \mathbb{C}^{(n-r) \times r}$  and  $A_3 \in \mathbb{C}^{(n-r) \times (n-r)}$  such that

$$B = U \begin{bmatrix} I_r & \vdots & A_1 \\ \cdots & \vdots & \cdots \\ A_2 & \vdots & A_3 \end{bmatrix} U^{-1}.$$

Denote by  $a^{(1)}, \dots, a^{(r)}$  the columns of  $A_2$ . Since

$$\text{rank}(A_2) \leq n - r = \dim \mathcal{N}(A) < \text{rank}(P) = r,$$

there is  $(\alpha_1, \dots, \alpha_r)^T \in \mathbb{C}^r$  such that  $(\alpha_1, \dots, \alpha_r) \neq 0$  and

$$\alpha_1 a^{(1)} + \dots + \alpha_r a^{(r)} = 0.$$

Set  $x = (\alpha_1, \dots, \alpha_r, 0, \dots, 0)^T \in \mathbb{C}^n$  and  $z = Ux$ , then  $z \neq 0$  and

$$Bz = U \begin{bmatrix} I_r & A_1 \\ A_2 & A_3 \end{bmatrix} x = Ux = z,$$

thus  $1 \in \sigma(B)$ . □

#### 4.2. Corollary.

(1)  $R_P = [0, \infty) \Leftrightarrow \dim \mathcal{N}(P) \geq \text{rank}(P)$ .

(2)  $R_P = [1, \infty) \Leftrightarrow \dim \mathcal{N}(P) < \text{rank}(P)$ .

*Proof.* (1) Theorem 4.1 and Corollary 2.2. (2) Theorem 4.1 and Proposition 1.2 (3). Observe that  $P \in \mathcal{G}_1(P)$  and  $r(P) = 1$ . □

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